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High-Order Fluctuation-Splitting Schemes for Advection-Diffusion Equations

Hiroaki Nishikawa and Philip Roe*

Department of Aerospace Engineering,
University of Michigan, Ann Arbor, MI 48109-2140
hiroakin@engin.umich.edu, philroe@engin.umich.edu

1 Introduction

Motivated by the failure of current finite-volume(FV) codes in accurately predicting hypersonic heat transfer rate on unstructured grids[1], we have been studying schemes based on multidimensional upwinding, or fluctuation-splitting(FS), as a radical alternative. This scheme is based on nodal variables and cell residuals (fluctuations): the latter drive the change of the former in a multidimensional fashion. The focus has been on its high-order extension with the incorporation of viscous terms, and primarily in the simplified context of two-dimensional advection-diffusion problems,

$$u_t + a\,u_x + b\,u_y = \nu\,(u_{xx} + u_{yy}). \tag{1}$$

In previous work, we found that a problem arises in regions where advection and diffusion effects are equally important (such a region always exists in the middle of a boundary layer): schemes for advection and diffusion cannot simply be added, or the scheme reduces to only 1st-order accuracy [2]. To avoid this and achieve high-order, we proposed to write the equation (1) as a first-order system(FOS) introducing gradients as new variables, and developed uniformly high-order FS schemes for the advection-diffusion equation. This approach was then followed also in developing high-order FS diffusion schemes on P_2 elements [3]. In this paper, we give a more detailed account for this strategy based on a conventional design principle for high-order methods, and show that the FOS approach has several advantages over a naive method.

2 Fluctuation-Splitting Schemes

Consider solving conservation laws of the form

$$u_t + f_x + g_y = 0 \tag{2}$$

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in a domain divided *irregularly* into a set of triangles $\{T\}$. Storing solutions at nodes $\{j\}$, we begin by computing for all triangles $T \in \{T\}$ the fluctuation ϕ^T which is a numerical approximation of the flux balance over the element

$$\phi^T = -\int_T (f_x + g_y)\ dxdy \tag{3}$$

evaluated by quadrature. If this vanishes, we have the best possible solution and take no action, but if it does not, we proceed on to distribute it, in a way that reflects multidimensional physics (upwind for advection; isotropically for diffusion), to the nodes. As a result, we obtain the following semi-discrete equation at each node.

$$\frac{du_j}{dt} = \frac{1}{M_j} \sum_{T \in \{T_j\}} \phi_i^T \qquad \forall j \in \{j\} \tag{4}$$

where $\{T_j\}$ is a set of triangles that share node j, M_j is the median dual cell area, and $\phi_j^T = \beta_j^T \phi^T$: β_j^T is the distribution coefficient that assigns the fraction of the fluctuation sent to node j in triangle T. If β_j^T is bounded, zero fluctuation implies no updates in the solution: the scheme preserves polynomial solution (depends on the quadrature used) on *arbitrary* unstructured grids; this is called the residual property and is part of the reason for the reduced mesh sensitivity of FS schemes. In this work, we are interested only in steady-state solutions and so we march in time simply using the forward Euler time integration until solutions do not change.

For the advection-diffusion equation where $(f, g) = (au - \nu u_x,\ bu - \nu u_y)$, the fluctuation ϕ^T is the sum of two parts, advection ϕ_a^T and diffusion ϕ_d^T. The former should be distributed with upwinding while the latter should be distributed isotropically. But they should not be distributed separately, or the residual property is lost and more importantly the accuracy of the scheme reduces to 1st-order [2]. In this work, we therefore consider only the total fluctuation, and distribute it with a combined upwind-isotropic distribution coefficient proposed in [2] which becomes upwind (the LDA scheme) in the advection limit and isotropic in the diffusion limit. This distribution coefficient is bounded, and therefore the residual property is guaranteed: the accuracy of the FS scheme is now determined by the accuracy of the fluctuation [4].

3 Second-Order Fluctuations

For conservation laws, the fluctuation can be evaluated as a contour integral

$$-\int_T (f_x + g_y)\ dxdy = -\oint_{\partial T} (f\,dy - g\,dx) \tag{5}$$

This is convenient because only the solution variation along the element boundary is relevant to the accuracy of the fluctuation. Consider a single term over an edge, say from node 1 to 2, $\int_1^2 f\,dy$. For second-order accuracy, it suffices to use the trapezoidal rule

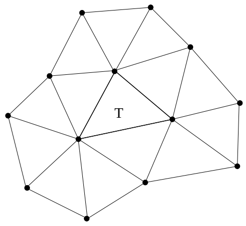

Fig. 1. DGE approach

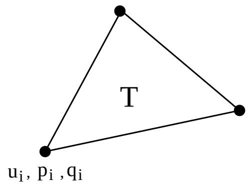

Fig. 2. FOS approach

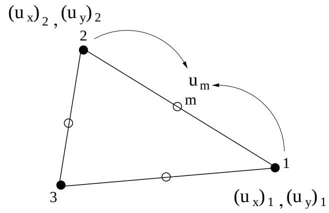

Fig. 3. High-Order

$$\int_1^2 f\, dy = \frac{f_1 + f_2}{2}(y_2 - y_1) + \mathcal{O}(h^3) \tag{6}$$

This formula is 2nd-order accurate provided the nodal fluxes f_1 and f_2 are also second-order accurate. Note that for the advection-diffusion equation the flux now involves the solution gradient u_x which is not available at nodes and must be evaluated with sufficient accuracy. In other words, we must follow the principle that the quadrature formula and the fluxes at the quadrature points must be of comparable accuracy, in the same spirit of other high-order methods [5, 6, 7]. This has not been stressed so much in FS methods for hyperbolic problems, where the solution values are stored at the quadrature points, and the condition is automatically met, but becomes important for problems such as advection-diffusion. The task becomes one of evaluating the diffusive fluxes (gradients) with sufficient accuracy *at the quadrature points*. In this paper we compare, theoretically and experimentally, two approaches. One of these is interpolation, and the other is solution of a first-order system having the gradient values as unknowns. The first of these is explicit, but the second is implicit.

One approach is to directly obtain the gradients by reconstruction, for example by a least-squares(LS) quadratic reconstruction which is 2nd-order on general triangular grids. Then, the fluctuation will have a residual property, preserving exact linear solutions. Note that although the diffusion part is exact for quadratic functions but it is only 2nd-order because both the quadrature and the recovered gradients are 2nd-order (being exact for quadratic does not imply 3rd-order accuracy for terms involving 2nd-order derivatives!). This is probably the simplest approach to achieve 2nd-order accuracy, but requires the stencil to be enlarged (Figure 1), even more so for higher-order as at least cubic reconstruction is needed beyond $\mathcal{O}(h^2)$. In the rest of the paper, we refer to this approach as the direct gradient evaluation(DGE) approach.

An alternative is the FOS approach in which we compute the nodal gradients as additional unknowns, solving the equivalent first-order system

$$u_t + f_x + g_y = 0, \ p - u_x = 0, \ q - u_y = 0 \tag{7}$$

where $(f, g) = (au - \nu p, bu - \nu q)$. Fluctuations are defined for the additional equations with piecewise linear variation of u, p, q, and then distributed isotropically to update p, q; this minimizes those fluctuations in an L_2 norm. Values of p and q computed this way are used to evaluate the diffusive part of the flux. This allows us to keep the stencil compact (Figure 2). The resulting scheme has the residual property that exact linear solutions are preserved. Observe that this FOS approach does not require any solution reconstruction and the scheme remains compact.

Incidentally, it is possible and interesting to interpret the FOS approach as a reconstruction method. In fact, the method is solving a globally coupled system for p and q in an iterative manner. Interestingly enough, if we invoke ‘mass-lumping’ to remove the coupling, we end up with the familiar Green-Gauss formula at every node. A similar observation is made in [8] where a global system is derived by the Galerkin method. Certainly, the mass-lumping yields a simple explicit formula for gradients, but in FS schemes we iterate toward a steady-state anyway, and so we may as well iterate for gradients along the way. Naturally, without mass-lumping, gradients are more accurate.

4 High-Order Fluctuations

For higher-order accuracy, we introduce a virtual node, m, at the midpoint over the edge, and evaluate the fluctuation by Simpson’s rule (Figure 3),

$$\int_1^2 f\,dy = \frac{f_1 + 4f_m + f_2}{6}(y_2 - y_1) + \mathcal{O}(h^5) \tag{8}$$

This integrates cubic polynomials exactly and therefore can be 4th-order accurate. To match the accuracy, we now need 4th-order accurate fluxes at nodes including the midpoint. This requires not only gradients at all points for the diffusive flux but also the midpoint value u_m for the advective part of f_m; all with 4th or at least 3rd-order accuracy to give a high-order scheme.

To estimate u_m with high-order, we follow Caraeni and Fuchs[9]. First recover the gradients $((u_x)_i, (u_y)_i)$ at nodes, and then evaluate u_m by the Hermite cubic interpolation along the edge (Figure 3).

$$u_m = \frac{u_1 + u_2}{2} - \frac{1}{8}\left\{(u_s)_2 - (u_s)_1\right\} \tag{9}$$

where $(u_s)_i \equiv (u_x)_i(x_2 - x_1) + (u_y)_i(y_2 - y_1)$. This formula is 3rd-order with quadratic gradient reconstruction and 4th-order with cubic reconstruction. This completes the advective part of the high-order fluctuation.

For the diffusive part, the DGE approach requires cubic/quartic reconstruction to ensure 3rd/4th-order accuracy. The reconstruction stencil will then become very large, but can be reduced somewhat by utilizing the midpoint values u_m already recovered for the advective part. For example, a node shared by 4 triangles has now 12 neighbor nodes (instead of 4) which enables cubic reconstruction. But this means that we must prepare two reconstruction algorithms: one for estimating u_m in the advective flux; the other for evaluating the diffusive flux. Also, we would need to store the midpoint values.

In contrast, the FOS approach can be extended to high-order without such elaboration. All we need are high-order accurate gradient variables (p, q) at nodes including the midpoints. The midpoint values can be estimated in exactly the same way as u_m. Then, with midpoint values available, we can define high-order fluctuations for the slope equations with piecewise quadratic variation of u, p, q, and distribute these to compute p, q at the original nodes [3]. Clearly, this scheme has the residual property: preserving exact quadratic/cubic solutions with quadratic/cubic gradient recovery. Observe now that essentially we need only a single algorithm that estimates midpoint values via gradient recovery and the Hermite interpolation over edges, and we only need to call this for 3 times to estimate u_m, p_m and q_m. We point out also that this scheme can be made even simpler by substituting (9) into (8) to eliminate the midpoint values, resulting an element-wise high-order correction scheme [3], i.e. no need to store the midpoint values in practice.

5 Results

We show results for the advection-diffusion equation with the exact solution $u = -\cos(\pi\eta)\exp(0.5\xi(1 - \sqrt{1 + 4\pi^2\nu^2})/\nu)$ where $\xi = ax + by$, $\eta = bx - ay$, $\nu = 0.1$ and $(a, b) = (7, 4)$ in the square domain $[0, 1] \times [0, 1]$. This is a case where the advective and the dffusive terms are equally important and a scheme obtained by adding the Galerkin scheme to an advection scheme indeed loses its accuracy. We tested the DGE schemes with quadratic/cubic/quartic LS gradient reconstruction, and the FOS schemes with no reconstruction and quadratic/cubic LS reconstruction, using the inverse distance-squared weighting for all reconstructions[10]. We determined the order of convergence through a series of computations using three different unstructured grids having 441, 1681, and 6561 nodes. Figures 4 and 5 show the results ($\circ$: 2nd; $*$: 3rd; $\Box$: 4th). First, 2nd, 3rd, 4th-order accuracy for both types of schemes were confirmed. Second, the convergence of the 3rd-order schemes looks somewhat better than expected. This could be because of the combination of the 4th-order quadrature formula and 3rd-order fluxes. Finally, and remarkably, the FOS schemes generate much lower error levels than the DGE schemes (note that the figures are equally scaled); this may be because the gradients in the flux are more accurate in the FOS approach as mentioned in Secion 3.

6 Concluding Remarks

The design principle of high-order methods equally applies to FS schemes: the accuracy is determined by the quadrature formula and the fluxes at quadrature points, and this is particularly important for the diffusive flux. With this in mind, we compared two different strategies (FOS and DGE) for solving the advection-diffusion equation, and conclude that the FOS approach demonstrate several advantages over the DGE approach. First, the FOS schemes are much more compact and accurate than the DGE schemes: no gradient recovery required for 2nd-order, and quadratic/cubic reconstruction for 3rd/4th-order. Second, high-order FOS schemes require only a single additional algorithm

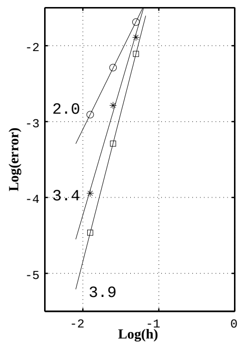

Fig. 4. DGE schemes

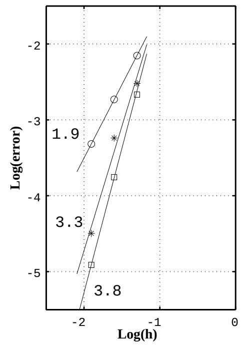

Fig. 5. FOS schemes

that estimates midpoint values via gradient recovery (and this can be done without storing the midpoint values[3]) whereas high-order DGE schemes add much more complications to coding. Also, the FOS approach provides a solid base for developing P_2-schemes (midpint values are computed as unknowns); preliminary studies are reported in [3]. Finally, we remark that in the FOS approach, only the gradients in the diffusive flux need to be stored (not gradients of all variables). In the Navier-Stokes equations, this means that we need to store 3 viscous stresses and 2 heat fluxes in 2D, and 6 viscous stresses and 3 heat fluxes in 3D.

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